On the incoherence of noiselet and Haar bases

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Abstract:

Noiselets are a family of functions completely uncompressible using Haar wavelet analysis. The resultant perfect incoherence to the Haar transform, coupled with the existence of a fast transform has resulted in their interest and use as a sampling basis in compressive sampling. We derive a recursive construction of noiselet matrices and give a short matrix-based proof of the incoherence.

1. Introduction

The noiselet basis, originally described in [2], has garnered interest recently because noiselets (1) are maximally incoherent to the Haar basis and (2) have a fast algorithm for their implementation. Thus, they have been employed in compressive sampling to sample signals that are sparse in the Haar domain [1].

The work presented here was motivated by the observation that it had not been previously shown in a straightforward way that the discrete Haar transform is maximally incoherent to a discretized version of the noiselet transform. Additionally, the exact form of a noiselet matrix needed to be inferred from the original work.

The main contributions are the derivation of a recursive, tensor product-based, construction of noiselet matrices, the unitary matrices that result from the noiselet transform for discrete input, and an intuitive proof showing its incoherence to the corresponding Haar matrix.

2. Preliminaries

2.1 General definitions

Definition 1. *Let A be an $m \times n$ matrix, and B be a matrix of an arbitrary size. The Kronecker product of A and B is*

$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix}.$$

The Kronecker product (see e.g. [4]) is a bilinear and associative operator which is not generally commutative. It can be combined with a standard maxtrix multiplication as follows:

$$(A \otimes B)(C \otimes D) = AC \otimes BD$$

whenever the products AC, BD exist. This property is sometimes called the *mixed product property*.

Definition 2. *Let A be a $m \times n$ matrix. $A(k,*)$ denotes the (row) vector $(A(k,1) \quad A(k,2) \quad \ldots \quad A(k,n))$ while, $A(*,l)$ similarly denotes the (column) vector $(A(1,l) \quad A(2,l) \quad \ldots \quad A(m,l))^T$.*

2.2 Noiselets

Noiselets [2] are functions that are completely uncompressible under the Haar transform. The family of noiselets is constructed on the interval $[0,1)$ as follows:

$$\begin{aligned} f_1(x) &= \chi_{[0,1)}(x), \\ f_{2n}(x) &= (1-i)f_n(2x) + (1+i)f_n(2x-1) \\ f_{2n+1}(x) &= (1+i)f_n(2x) + (1-i)f_n(2x-1) \end{aligned}$$

Here, $\chi_{[0,1)}(x) = 1$ on the definition interval $[0,1)$ and 0 otherwise. It is shown in [2] that $\{f_j\}$ is a basis:

Theorem 1. *The set $\{f_j | j = 2^N, \ldots, 2^{N+1}-1\}$ is an orthogonal basis of the vector space V_{2^N}, which is the space of all possible approximations at the resolution 2^N of functions in $L^2[0,1)$.*

2.3 Haar Transform

Haar wavelet transform can be described by a real square matrix. For our purposes, it is advantageous to recursively build the Haar matrix using the Kronecker product [3]:

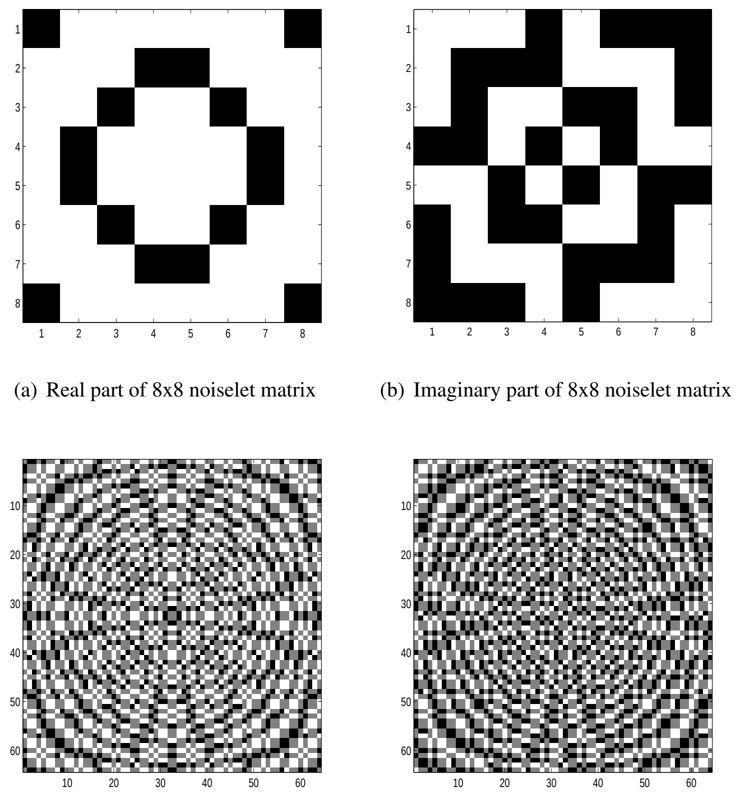
$$H_n = \frac{1}{\sqrt{2}} \begin{bmatrix} H_{n/2} \otimes (1 \quad 1) \\ I_{n/2} \otimes (1 \quad -1) \end{bmatrix}.$$

The iteration starts with $H_1 = [1]$. The normalization constant $\frac{1}{\sqrt{2}}$ ensures that $H_n^T H_n = I$. Haar wavelets are the rows of H_n.

3. Matrix construction of noiselets

First we extend and discretize the noiselet functions.

Definition 3. *The extensions of noiselets to the interval $[0, 2^m - 1]$ sampled at points $0, \ldots, 2^m - 1$ is the series*



(a) Real part of 8x8 noiselet matrix (b) Imaginary part of 8x8 noiselet matrix

(c) Real part of 64x64 noiselet matrix (d) Imaginary part of 64x64 noiselet matrix

Figure 1: Noiselet matrix: graphical view. In figures (a) and (b), the black and white colors denote values of -0.25 and 0.25 respectively. In figures (c) and (d), the black, gray and white colors denote values of -0.125, 0 and 0.125 respectively. .

of functions $f_m(k,l)$

$$f_m(1,l) = \begin{cases} 1 & l = 0,\dots,2^m - 1 \\ 0 & \textit{otherwise} \end{cases}$$

$$f_m(2k,l) = (1-i)f_m(k,2l) + (1+i)f_m(k,2l-2^m)$$

$$f_m(2k+1,l) = (1+i)f_m(k,2l) + (1-i)f_m(k,2l-2^m)$$

where m *denotes the range of extension,* $k = 1,\dots,2^{m+1}$ *is the function index and* $l = 0,\dots,2^m - 1$ *is the sample index.*

Starting with a 1×1 matrix N_1, a sequence of noiselet matrices $N_1, N_2, N_4, \dots, N_{2^m}$ of sizes 1×1, 2×2, 4×4, $\dots$, $2^m \times 2^m$, respectively, is generated. The rows of the N_n matrix are noiselets which form an orthonormal basis for the space $\mathbb{C}^n$.

Definition 4. *For* $n = 1$, $N_1 = \begin{bmatrix}1\end{bmatrix}$. *Then the* $n \times n$ *noiselet matrix* N_n *is built up recursively according to:*

$$N_n(k,*) = \frac{1}{2}(1-i \quad 1+i) \otimes N_{n/2}(\frac{k}{2},*)$$

when $k = 0,2,4,\dots,n-2$ *and*

$$N_n(k,*) = \frac{1}{2}(1+i \quad 1-i) \otimes N_{n/2}(\frac{k-1}{2},*)$$

when k=1,3,. . . ,n-1.

Lemma 1. *Let* $m > 0$. *The noiselet matrices* $N_1, N_2, N_4, \dots, N_{2^m}$ *are built up from a series of discretised and extended noiselets* f_m*:*

$$N_n(k,l) = f_m(n+k, \frac{2^m}{n}l), \quad k,l = 0,\dots,n-1.$$

Proof. Let $m > 0$ be fixed. For $n = 1$

$$N_1(0,0) = f_m(1,0) = 1.$$

By induction, for a matrix of size $n = 2^p, p = 1,\dots,m$, its basis vector $k = 0,2,4,\dots,n-2$ and vector indices $l = 0,\dots,\frac{n}{2}-1$

$$\begin{aligned} N_n(k,l) &= (1-i)N_{n/2}(\frac{k}{2},l) \\ &= (1-i)f_m(\frac{n}{2}+\frac{k}{2},\frac{2^m}{\frac{n}{2}}l) = f_m(n+k,\frac{2^m}{n}l). \end{aligned}$$

For the same n, k and $l = \frac{n}{2},\dots,n-1$,

$$\begin{aligned} N_n(k,l) &= (1+i)N_{n/2}(\frac{k}{2},l-\frac{n}{2}) \\ &= (1+i)f_m(\frac{n}{2}+\frac{k}{2},2\frac{2^m l}{n}-2^m) = f_m(n+k,\frac{2^m}{n}l). \end{aligned}$$

To see this, observe that f_m is zero outside of $[0,2^m-1]$ and therefore, the first half of samples of $f_m(k,l)$ are defined exclusively by the expression $(1 \pm i)f_m(k,2l)$

whereas the second half of the samples are defined exclusively by $(1 \pm i) f_m(k, 2l - 2^m)$.
For k odd ($k = 1, 3, \ldots, n-1$) the proof is similar. $\square$

Specially, the noiselet matrix N_n for $n = 2^m$ can be found as the "tail" of the function series f_m. Indeed, the expression in Theorem 1 becomes $N(k,l) = f_m(n+k, l)$ for $n = 2^m$.

4. Incoherence of noiselets and Haar

In what follows, we adhere to the terminology of basis coherence which is common in the field of compressive sampling. See for example [1] for details on these definitions and related literature.
Mutual coherence of two bases is defined as the maximum scalar product of any pair of their basis vectors:

Definition 5. *Mutual coherence between two orthonormal bases Ψ, Φ is*

$$\mu(\Psi, \Phi) = \max_{k,j} |\langle \psi_k, \phi_j \rangle|.$$

The minimal coherence is usually termed *maximal* or *perfect* incoherence, which means that $\mu(\Psi, \Phi) = O(1)$. In other words, the matrix of scalar products $\Psi\Phi^*$ is "flat". As Candès and Romberg suggest [1], we will show the perfect incoherence of Haar and noiselets in the following setting. Given an orthonormal $n \times n$ Haar matrix H, we compute the matrix of scalar products for a corresponding noiselet matrix N normalized such that $N^*N = nI$. By doing so, the product will be flat with all values having the magnitude of 1.
For clarity of the main proof, it saves some technical work to define a "twisted" noiselet basis.

Definition 6. *The twisted noiselet matrix $\hat{N}_1 = \begin{bmatrix}1\end{bmatrix}$. Then the $n \times n$ twisted noiselet matrix $\hat{N}_n$ is built up recursively by*

$$\hat{N}_n(k, *) = \frac{1}{2} \hat{N}_{n/2}(\frac{k}{2}, *) \otimes (1-i \quad 1+i)$$

when $k = 0, 2, 4, \ldots, n-2$ and

$$\hat{N}_n(k, *) = \frac{1}{2} \hat{N}_{n/2}(\frac{k-1}{2}, *) \otimes (1+i \quad 1-i)$$

when $k = 1, 3, \ldots, n-1$.

The difference between this and the definition of the noiselet matrix N (Definition 4) is that the order of operands in the Kronecker product is changed. In fact, each one is just a permutation of the other.

Lemma 2. *For $n = 2^m$, the bases N_n, $\hat{N}_n$ consist of the same set of basis vectors.*

Proof. Indeed, we can write $\hat{N}_n = P_n N_n$ where P is the permutation matrix:

$$P(k, *) = \begin{cases} (1 \quad 0) \otimes P_{n/2}(\frac{k}{2}, *) & k = 0, 2, 4, \ldots, n-2 \\ (0 \quad 1) \otimes P_{n/2}(\frac{k-1}{2}, *) & k = 1, 3, \ldots, n-1 \end{cases}$$

starting with $P = [1]$.

The claim holds for $n = 1$. For $n = 2, 4, 8, \ldots, 2^m$,

$$P_n N_n(k,l) = P_n(k, *) N_n(l, *)^T$$

as it can easily be shown that N_n is symmetric. Using the recurrent equations for P_n and N_n and applying the mixed product rule, we get, for $k = 0, 2, 4, \ldots, n-2$,

$$P_n N_n(k,l) = \frac{1}{2}(1-i) P_{n/2}(\frac{k}{2}, *) N_{n/2}(*, \frac{l}{2})$$

when $l = 0, 2, 4, \ldots, n-2$ and

$$P_n N_n(k,l) = \frac{1}{2}(1+i) P_{n/2}(\frac{k-1}{2}, *) N_{n/2}(*, \frac{l}{2})$$

when $l = 1, 3, \ldots, n-1$. By induction,

$$P_n N_n(k, *) = \frac{1}{2} \hat{N}_{n/2}(\frac{k}{2}, *) \otimes (1-i \quad 1+i)$$

for even k indices. This situation for odd k is similar. $\square$

Now the main result can be shown.

Theorem 2. *Let $n = 2^m$ where m is a non-negative integer. Let N_n be the noiselet matrix of size $n \times n$ and let H_n be the Haar matrix of size $n \times n$. Then H_n and N_n are maximally incoherent.*

Proof. Without loss of generality, assume the bases are normalized such that $H_n^T H_n = I$ and $N_n^* N_n = nI$. For the case of $n = 1$,

$$H_1 N_1^* = \begin{bmatrix}1\end{bmatrix} \cdot \begin{bmatrix}1\end{bmatrix} = \begin{bmatrix}1\end{bmatrix}$$

For $n = 2^m, m > 1$, the incoherence is shown by induction. Suppose we know maximal incoherence holds for $\frac{n}{2}$ and we want to show it for n. In the induction step, we use the iterative construction of the Haar matrix by means of Kronecker product. By computing the product

$$H_n \hat{N}_n^* = H(N_n^* P_n^*) = (H_n N_n^*) P_n^T$$

we will still be able to conclude on magnitude of the elements of $(H_n N_n^*)$, since permutation matrices do not change magnitudes.
The product $H_n \hat{N}_n^*$ can be computed per-column; we take the j-th column of $\hat{N}_n^*$, $j = 0, 2, 4, \ldots, n-2$ and transform it by H_n, getting

$$\begin{aligned} H_n \hat{N}_n^*(*, j) = & \frac{1}{\sqrt{2}} \begin{bmatrix} H_{n/2} \otimes (1 \quad 1) \\ I_{n/2} \otimes (1 \quad -1) \end{bmatrix} \\ & \cdot \frac{1}{\sqrt{2}} \hat{N}_{n/2}^*(*, \frac{j}{2}) \otimes (1-i \quad 1+i)^* \end{aligned}$$

Note the altered normalization factor of noiselets. Now the mixed product property can be applied to get

$$\begin{aligned} \frac{1}{2} \begin{bmatrix} H_{n/2} \hat{N}_{n/2}^*(*, \frac{j}{2}) \otimes (1 \quad 1) \begin{bmatrix} 1+i \\ 1-i \end{bmatrix} \\ I_{n/2} \hat{N}_{n/2}^*(*, \frac{j}{2}) \otimes (1 \quad -1) \begin{bmatrix} 1+i \\ 1-i \end{bmatrix} \end{bmatrix} = \\ \frac{1}{2} \begin{bmatrix} H_{n/2} \hat{N}_{n/2}^*(*, \frac{j}{2}) * 2 \\ I_{n/2} \hat{N}_{n/2}^*(*, \frac{j}{2}) * 2i \end{bmatrix}. \end{aligned}$$

By induction, it follows that $|H_{n/2} \hat{N}_{n/2}^*(i, \frac{j}{2})| = 1$ and $|I_{n/2} \hat{N}_{n/2}^*(i, \frac{j}{2})| = 1$ for $i = 1, \ldots, \frac{n}{2}$. The Kronecker multiplication is only by entries with magnitude 2, thus the resulting magnitudes are $\frac{1}{2} * 2 = 1$. The proof is equivalent for $j = 1, 3, \ldots, n-1$. $\square$

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