Wavelet Theory and Applications: Singapore, August 2004

University of Wisconsin – Madison

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The mathematical theory of pyramidal algorithms

CAP representations
Outline

The pyramidal algorithm of Burt and Adelson

Wavelet and framelet pyramids

Function space characterizations via wavelets and framelets

Approximation properties of framelets

Compression-Alignment-Prediction (CAP) representations and their use in function space characterizations

Compression-Alignment-Modified Prediction (CAMP) representations and their use in function space characterizations

The pyramidal algorithm of Burt and Adelson

Coefficients
Thomas Hangelbroek, Sangnam Nam, Jeil Kline, Steven Parker.

2nd row: Julia Velikina, Youngmi Hur, Yeon Kim, Nanth Stefnsson.

From left, 1st row: Wisconsin
Julia Velikina: undersampled MRI data
Approximate recovery of new data representation in NMR
Adaptive framelet-based representation of a vibraphone recording.
Nath Stefnsson: sparse framelet representations
6/10  61440 coefficients

box15, box17, box18  35085 coefficients

cubic spline  34608 coefficients

quartic spline  34452 coefficients

6/10  61440 coefficients
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Manage Data: Only data shown is the data shown.
Pyramidal algorithms (Burt and Adelson, 1983)

$P$ is prediction or subdivision

$\downarrow_{\mathcal{P}} \ast y =: \mathcal{P}_D \approx \downarrow_{\mathcal{P}} y$

$y$ is then predicted from $\mathcal{P}_D$

$C$ is compression or Coarsification

$\downarrow_{\mathcal{C}} \uparrow_{\mathcal{C}} (\downarrow_{\mathcal{C}} \ast y) =: \downarrow_{\mathcal{C}} \uparrow_{\mathcal{C}} C = \downarrow_{\mathcal{P}} y$

$\forall z \in \mathbb{Z} \ni C \subseteq \mathbb{Z} \ni \downarrow_{\mathcal{C}} \uparrow_{\mathcal{C}} y$

oriented $\mathbb{Z} \ni (\downarrow_{\mathcal{C}} \ast y)$

$\downarrow_{\mathcal{P}} y = 1 = (\gamma) y \mathbb{Z} \ni y = (\gamma) y$

$y$ is a symmetric, normalized, filter:

$h(k) = h(1-k)$, $P_k \in \mathbb{Z}$

$h(k) = 1$.

$y$ is a downsampling $\downarrow$, upsampling $\uparrow$
The pyramid algorithm:

Define the detail coefficients:

\[ d^j:=(I PC)y^j \]

Replace \( y^j \) by the pair \( (y^j, d^j) \). Continue iteratively.

Reconstructing \( y^m \) from \( y^0, d^1, d^2, \ldots, d^m \) is trivial:

\[ y^m = d^m + Py^{m-1} \]

\[ y^{m-1} = d^{m-1} + Py^{m-2} \]

\[ \vdots \]

\[ y^1 = d^1 + Py^0 \]

Reconstruction. Recovering \( y^m \) from \( y^0, d^1, d^2, \ldots, d^m \) is trivial.
Wavelet pyramids, Mallat, 1987

Decompose the detail map \( \Phi \) by passing the signal \( y \) through a real, symmetric, highpass filter \( h_1 \) with \( h_1 \), and so on.

Note that we can recover \( y_m \) from \( y_0, \ldots, y_m, \), since

\[
\begin{align*}
\downarrow & y, \quad y^\ast y, \\
0 = (y, y) \text{ with } y, \text{ a real, symmetric, highpass filter,}
\end{align*}
\]

Then

\[
D \Phi = \mathcal{C}p - I
\]

Decompose $I - \mathcal{P} \mathcal{D}$ where

Framelets Pyramids, Daubechies-Han-R-Shen, 03

\[ \mathcal{P} \mathcal{D} \subseteq \mathcal{C} \]

Who needs the overhead associated with framelets?

A: They offer far greater design freedom

\[ \mathcal{P} \mathcal{C} = 1 \]

Each $\eta$ real, (anti-)symmetric, highpass:

\[ (\downarrow \hat{\eta} \ast \hat{\eta}) \leftarrow \hat{\eta} : \eta \]

where

\[ (\downarrow \hat{\eta} \ast \hat{\eta}) \leftarrow \hat{\eta} : \eta \]

\[ \mathcal{D} \]

\[ \mathcal{D} \]

where

\[ \mathcal{D} \]

\[ \mathcal{D} \]

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\[ \mathcal{D} \]
Piecewise linear

All the filters above are 3-tap, and the underlying wavelets are

$$b_i = 4 + e_i$$

$$c_{h1} = p_1^4 + e_2 i$$

$$c_{h2} = 1 + 4 e_i$$

Refinable function

Mother wavelet 1

Mother wavelet 2

Piecewise-linear

RS2

$$z \left( \frac{z^2 - \alpha - 1}{1} \right) \frac{p}{z} = (z) y$$

$$z \left( \frac{z^2 - \alpha - 1}{1} \right) \frac{p}{z} = (z) y$$

$$z \left( \frac{z^2 - \alpha + 1}{1} \right) \frac{p}{z} = (z) y$$

Example (R-Shen, 1997)
The mathematics behind pyramidal algorithms, Part I: the rudiments

Given $h$, one looks for $\phi \in L_2(\mathbb{R})$ s.t.

$$\hat{\phi}(2^j) = \hat{h}\hat{\phi}, \quad (\hat{\phi}(0) = 1).$$

(1)

$\phi$ is a refinable function, the filter $h$ is the (lowpass) refinement mask.

Notation: For $j, k \in \mathbb{Z}$, $\varphi_{j,k} := 2^{j/2}\varphi(2^j \cdot k)$. ($\varphi$ some function)

Then $y_{j,f} = Cy_{j+1,f}$, $\forall j$.
themathematicsbehindpyramidalalgorithms,
PartII:wavelet-basedcharacterizationsoffunctionspaces

let \( L^2(\mathbb{R}) \setminus L^1(\mathbb{R}) \) s.t. \( R(t)dt = 0 \)

Waveletsystem \( X(t) \) is (orthonormal)

\[ \{ \mathbb{Z} \ni \xi, \zeta : (\chi - \chi) \phi_{\xi, \zeta} = \chi \ } \] \( ^{\circ} \phi \} =: (\phi)X \)

Whywavelets?

basis of \( T^2(\mathbb{R}) \).

Wavelet system \( X(t) \) is (orthonormal)

\[ 0 = \mathbf{p}(t) \phi \int_{-\infty}^{\infty} (\mathbb{R}) \frac{1}{T} \bigcup \mathbb{R}^2 T \ni t \cup \mathbb{R}^2 T \ni t \in \phi \]

functions

functions

Part II: wavelet-based characterizations of

The mathematicis behind pyramidal algorithms,
The function space $L_\text{sp}$ is the set of all $f \in \mathcal{S}$ such that $\supp f' \in \mathcal{S}$ and $m \in \mathbb{N}$ and $\infty > d > 1$. The Sobolev space $H^d_{w,M}$ is $L^d_w \cup H^0_0$. The Hardy space $H^d$ is $L^d_0$. The function space $L^d_0$ is the set of all $f$ such that $d > 0$ and $m \in \mathbb{N}$. Let $\mathcal{S} \subset \phi$ satisfy $(\infty > d > 0, m \in \mathbb{N}) \in L^d_0$. Function spaces
Theorem 1 (Meyer, Frazier-Jawerth, 198x)

\[ t > t \geq 0 \quad \left\{ \begin{array}{c} 0 \vspace{1pt} \text{otherwise} \\ 1 \end{array} \right. \}

=: (t)X \left( \begin{array}{c} \chi_{s} \{ \cdot \} \\
\int_{\mathbb{R}} \chi_{s} \{ \cdot \} \langle \chi_{s} \xi \phi, f \rangle \end{array} \right) =: f \ast \mathcal{O}

where

\[ d \int \| f \ast \mathcal{O} \| \approx d \int \| f \| \]

Then we have

\[ 1 - \nu \geq \alpha \geq 0 \quad \forall 0 = \nu \quad \int^{C} \mathcal{O} \subset \phi \]

\[ \text{is orthonormal wavelet, and:} \]

\[ \{ s : s - d/1, \max \{ s' - s, s' - s, s' - s \} \} \]

Theorem 1 (Meyer, Frazier-Jawerth, 198x)

Characterization of L using wavelets
\[ X \text{ is a non-redundant tight frame } \iff \{ \Phi \} X \]

\[
\Phi \subset \mathbb{L}^2 \quad \text{if } \forall f \in \mathbb{L}^2 \exists \Phi \subset \mathbb{L}^2 : \langle \Phi, f \rangle = \langle \Phi, f \rangle
\]

The analysis map is the map

\[
\{ \forall \Phi \subset \mathbb{L}^2 : (\Phi \cdot \mathbb{L}) \Phi = \Phi \} =: (\Phi) X
\]
Functions spaces

Part III: Framelet-based characterizations of

The mathematics behind pyramidal algorithms.

\[ \text{Bad News: (2) is too stringent.} \]

\[ \Phi \subset \Phi \subset \Phi \subset \Phi \]

\[ \text{where} \quad \| f \|_\Phi \approx \| f \|_\Phi \]

\[ \Phi \subset \Phi \subset \Phi \subset \Phi \]

\[ \text{Theorem 2 (Kyriazis, Nielsen)} \]

\[ \Phi \subset \Phi \subset \Phi \subset \Phi \]

\[ \text{Integer:} \quad \{ s - d/k, s - d/k, \} \max < u, \infty > d > 0, \Xi \subset \Xi \]

\[ \text{Part III: Framelet-based characterizations of} \]

\[ \text{The mathematics behind pyramidal algorithms.} \]
Construction of wavelets and framelets

1. Choose an analytic function \( \eta \) with refinement (lowpass) filter \( h \).

2. \( V_0 := \text{closed linear span of} \{ \varphi_k \} \quad \text{for} \ k \in \mathbb{Z} \).

3. Choose mother wavelets \( \{ \varphi_1, \ldots, \varphi_r \} \) from the framelet pyramids are the filters of \( \varphi_k \).

Then \( X(\Phi) \) is a tight frame (framelet).

\[
\left\{ \begin{array}{ll}
\nu = n, & \text{if } \nu = 0, \\
0, & \text{if } 0 = n, \end{array} \right. \quad \left( \begin{array}{l}
\nu = n, \\
I = 1
\end{array} \right) = \left( \begin{array}{l}
(n + \cdot) \quad \mathbb{Z} \\
\bigcup
\end{array} \right) + \left( n + \cdot \right) \eta
\]

Theorem 3 (R.-Shen, 1997)

Assume \( h \) as an orthonormal wavelet construction:

\[
(\varphi, \cdot) = \langle \eta - \cdot \rangle \quad \text{closed linear span of} \ \
(\varphi_0) \quad \text{is a tight frame (framelet)}.
\]

Then \( X(\Phi) \) is a tight frame (framelet).

\[
\left\{ \begin{array}{ll}
\nu = n, & \text{if } \nu = 0, \\
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(n + \cdot) \quad \mathbb{Z} \\
\bigcup
\end{array} \right) + \left( n + \cdot \right) \eta
\]

3. Choose mother wavelets \( \{ \varphi_1, \ldots, \varphi_r \} = \mathcal{W} \) and periodic \( \varphi_0 \).

Then \( \mathcal{W} \cap \mathcal{V} \) is closed linear span of \( (\varphi_0) \).

2. \( \varphi_0 \) is closed linear span of \( \varphi \) with the refinement (lowpass) filter \( \eta \).

1. Choose a renormalizable function \( \varphi \in L^2 \) with the refinement (lowpass) filter \( \eta \).

Construction of wavelets and framelets
How to measure the performance of framelets

1. For framelets, $m^0$ can be as small as $m'/2$.

2. For orthonormal wavelets, $m = m'$.

$$\{0^m, 2m\} = \min \{m\}$$

**Theorem 4 (Daubechies-Han-R-Shen, 2003)**

For orthonormal wavelets, $m = m^0 = m'$.

For framelets, $m^0$ can be as small as $m'/2$.

$$\text{dist}(\varphi; \mathcal{V}_n) = \min_{\varphi \in \mathcal{V}_n} \| \varphi - \mathcal{V}_n \varphi \|$$

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$$\text{dist}(\varphi; \mathcal{V}_n) = \min_{\varphi \in \mathcal{V}_n} \| \varphi - \mathcal{V}_n \varphi \|$$

Theorem 4 (Daubechies-Han-R-Shen, 2003)

For orthonormal wavelets, $m = m^0 = m'$.

For framelets, $m^0$ can be as small as $m'/2$.
The CAP operators are:

\[
\begin{align*}
\text{(Prediction-Subdivision)} & : \quad (\downarrow h) * \hat{d} y =: h_p \leftarrow h : P \\
\text{(Alignment)} & : \quad h * \hat{a} y =: h_A \leftarrow h : A \\
\text{(Coarsification-Compression)} & : \quad \uparrow (h * \hat{c} y) \leftarrow h : C
\end{align*}
\]

For all \( f \in \mathbb{Z} \), define \( \langle f \rangle \) via:

\[
\langle f \rangle \in \mathbb{Z}_2\{\phi \} = \langle g \rangle \in \mathbb{Z}_2\{\phi \}, \quad \text{for all } g \in \mathbb{Z}_2\{\phi \}
\]

Decompose: Fix \( f \in \mathbb{R} : f \leftarrow \mathbb{C} \).

A third (Auxiliary-Alignment) lowpass filter \( h_a \).

Two remaining functions \( h_c, h_p \).

Choose:

CAP Representations
The detail coefficients are:

\[ d_j = (A P A) y_j = Ay_j P A y_j 1 \]

\[ y_m = \text{CAP representation with } (d_j) \text{ the CAP coefficients}. \]

\[ \text{This is the CAP representation with } (f_p) \text{ the CAP} \]

\[ f_p (A P A) = f_p (A P A) = f_p \]

The detail coefficients are:
The mathematics behind pyramidal algorithms, Part IV: CAP-based characterizations of function spaces, or the winner takes all functions spaces.

Theorem 5

\[
\mathcal{O} \left( \sum_{j=1}^{2^{2d}} \chi_{s,f}(y)^{1+\frac{j}{p}} \right) =: f_{dAVsG}O
\]

where

\[
d_T \| f_{dAVsG}O \| \approx \frac{dT}{s} \| f \|
\]

Then:

\[
\left( \left\{ u^i \cdot v \right\}_{\text{max} \cdot \cdot \cdot} \right) O = \left( v + \cdot \right) y \left( s + \cdot \right) O
\]

and assuming that, \( d \varphi \subseteq \varphi \)

\[
\left. \left\{ s - I - d/1, s \right\} \right\}_{\text{max} \cdot \cdot \cdot} < \left. \varphi \right. \text{ integers, } u \cdot v \cdot d
\]

Theorem 5

Part IV: CAP-based characterizations of
Nowavelets, no framelets, zilch.
Wavelets are non-redundant. Caplets are only slightly redundant in high dimensions. Their redundancy is non-essential.

<table>
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<th>Avoid redundant representations</th>
<th>Have simple constructions</th>
<th>Very short filters, with no artifacts</th>
<th>Avoid mother wavelets</th>
<th>Provides good function space characterizations</th>
<th>Implemented by fast pyramid algorithms</th>
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**Summary**

Do they very short filters, with no artifacts?
With CAP in hand, one can modify the prediction process s.t.

\[ \text{Example: If } h \text{ is interpolatory, we may redefine the details as:} \]

\[ \mathcal{h}(\frac{y}{2}, (y)_{t+\ell}^{t+\ell+\ell} (c_d - I) \} =: (y)_{t+\ell}. \]
The performance is much better than Haar.

The filters for computing $\mathcal{F}$ are 4-tap on average; same as 2D Haar.

Reconstruction is as before, with a small tweak.

\[
\{D, \Lambda, H\} \in \mathcal{S}, \quad (f_{\mathcal{F}})^{s \in \mathcal{V}} - f_{\mathcal{F}} =: f_{p} \quad (\downarrow 1 - f_{\mathcal{F}}) - f_{\mathcal{F}} =: \dagger f_{p}
\]

Define the detail quadruplet as

\[
f_{p} := (f_{p}^{D}, f_{p}^{\Lambda}, f_{p}^{H}, f_{p}^{*})
\]

For each $f_{\mathcal{F}}$, let $\downarrow (\uparrow f_{\mathcal{F}}) =: f_{\mathcal{F}}^{*}$ and consider a partition $\mathcal{P}$ of $\mathcal{F}$ given as

\[
y := [0, \frac{8}{1}, \frac{8}{1}, \frac{8}{1}, \frac{8}{1}, \frac{8}{1}, \frac{8}{1}, \frac{8}{1}, 0] = dy = \omega y, \quad I = \mathcal{V}
\]

Example (2D):
First level $p$ CAMP coefficients organized by cosets.