Reminders: sampling and Fourier

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Reminders on analog signals

- Integrable signals

$$s \in L_1(\mathbb{R}) \longleftrightarrow \int_{-\infty}^{\infty} |s(t)| dt < \infty$$

- Finite energy signals

$$s \in L_2(\mathbb{R}) \longleftrightarrow ||s||^2 = \int_{-\infty}^{\infty} |s(t)|^2 dt < \infty$$

- Scalar product

$$\langle s_1, s_2 \rangle = \int_{-\infty}^{\infty} s_1(t) s_2^*(t) dt$$

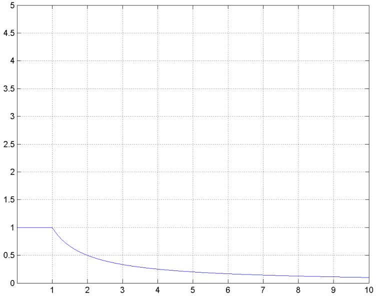
- Remark

$$L_1(\mathbb{R}) \not\subset L_2(\mathbb{R}) \quad \text{and} \quad L_2(\mathbb{R}) \not\subset L_1(\mathbb{R})$$

Reminders - Counter-example 1

- Signal in $L_2(\mathbb{R})$ but not in $L_1(\mathbb{R})$

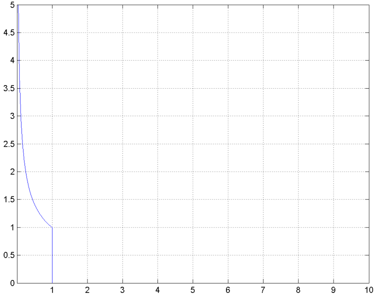
$$x = \begin{cases} 1 & \text{if } |t| < 1 \\ \frac{1}{|t|} & \text{otherwise.} \end{cases}$$



Reminders - Counter-example 1

- Signal $L_1(\mathbb{R})$ but not in $L_2(\mathbb{R})$

$$x = \begin{cases} \frac{1}{\sqrt{t}} & \text{if } |t| < 1 \\ 0 & \text{otherwise.} \end{cases}$$



Continuous Fourier transform

- Definition

$$s(t) \xrightarrow{\text{FT}} S(f) = \int_{-\infty}^{\infty} s(t) e^{-\imath 2\pi f t} dt$$

  function of the (dual) variable frequency $f \in \mathbb{R}$
- Existence 1: if $s \in L_1(\mathbb{R})$, then
  - S(f) is continuous and bounded
  - $\lim_{|f| \to \infty} S(f) = 0$
- Existence 2:

$$s \in L_2(\mathbb{R}), \text{ iff } S \in L_2(\mathbb{R})$$

- Inversion:

$$s(t) = \int_{-\infty}^{\infty} S(f) e^{\imath 2\pi f t} df$$

  for almost every t

Fourier transform - properties 1

- Linearity

$$s_1(t) \xrightarrow{\text{FT}} S_1(f), \quad s_2(t) \xrightarrow{\text{FT}} S_2(f)$$

then

$$\forall(\lambda, \mu) \in \mathbb{C}^2 \quad \lambda s_1(t) + \mu s_2(t) \xrightarrow{\text{FT}} \lambda S_1(f) + \mu S_2(f)$$

- Delay/translation

$$\forall b \in \mathbb{R}, \quad s(t-b) \xrightarrow{\text{FT}} e^{-\imath 2\pi b f} S(f)$$

delay/translation invariance in amplitude spectrum

- Modulation:

$$\forall \nu \in \mathbb{R}, \quad e^{\imath 2\pi \nu t} s(t) \xrightarrow{\text{FT}} S(f-\nu)$$

Fourier transform - properties 2

- Scale change

$$\forall \alpha \in \mathbb{R}^*, \quad s(\alpha t) \xrightarrow{\text{FT}} \frac{1}{|\alpha|} S(\frac{f}{\alpha})$$

turns dilatation onto contraction

- Time inverse: a special case

$$s(-t) \xrightarrow{\text{FT}} S(-f)$$

- Conjugation

$$s^*(t) \xrightarrow{\text{FT}} S^*(-f)$$

- Hermitian symmetry: if s is real, then:
  - $\mathrm{Re}(S(f))$ even, $\mathrm{Im}(S(f))$ odd
  - $|S(f)|$ even, $\arg S(f)$ odd $\pmod{2\pi}$

Fourier transform - properties 3

- Convolution

$$(s_1 * s_2)(t) = \int_{-\infty}^{\infty} s_1(u) s_2(t-u) du = (s_2 * s_1)(t)$$

- Conditions
  - $s_1 \in L_1(\mathbb{R}), s_2 \in L_1(\mathbb{R}) \Rightarrow s_1 * s_2 \in L_1(\mathbb{R})$
  - $s_1 \in L_2(\mathbb{R}), s_2 \in L_2(\mathbb{R}) \Rightarrow s_1 * s_2 \in L_2(\mathbb{R})$

$$(s_1 * s_2)(t) \xrightarrow{\text{FT}} S_1(f) S_2(f)$$

- Parseval-Plancherel equalities: if $s_1, s_2 \in L_1(\mathbb{R})$:
  - $\langle s_1, s_2 \rangle = \langle S_1, S_2 \rangle$
  - $\|s\|^2 = \|S\|^2$

Fourier transform - examples

- $s(t) = e^{-at}u(t), \Re(a) > 0$: $S(f) = \frac{1}{a+2\imath\pi f}$
- $f(t) = e^{-at}\cos(2\pi f_0 t)u(t)$:
  $S(f) = \frac{1}{2}\left[\frac{1}{a+2\pi\imath(f-f_0)} + \frac{1}{a+2\pi\imath(f+f_0)}\right]$
- $s(t) = e^{-|a|t}$: $S(f) = \frac{2|a|}{a^2+(2\pi f)^2}$

Digital signals - sampling

- For a signal $s(t)$, regularly sampled:

$$s[k] = s(kT)$$

with T: sampling period; $F = 1/T$: sampling frequency

- Sampling theorem: if $S(f) = 0$ for $|f| \geq B$ and $F \geq 2B$, the the signal is sampled without information loss (theoretically), with Shannon-Nyquist formula:

$$s(t) = \sum_{k=-\infty}^{\infty} s[k] \mathrm{sinc}\left(\frac{\pi(t-kT)}{T}\right), \mathrm{sinc}(t) = \begin{cases} \frac{\sin(t)}{t} & \text{if } t \neq 0 \\ 1 & \text{otherwise.} \end{cases}$$

- Note: Balian-Low theorem for time-frequency analysis

Digital signals - sampling

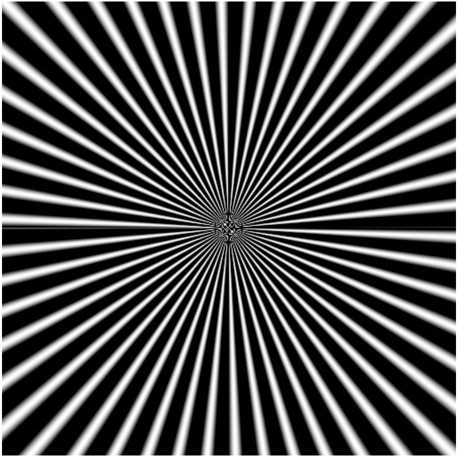


Figure 1: Example of aliasing

Digital signals - sampling

- Comments
  - sufficient but non-necessary condition
  - two-part theorem: sampling/reconstruction
  - jitter, amplitude quantization, noise
  - slow convergence of the sinc, instability
  - signals cannot be time-limited and frequency-bounded (together)
  - extensions to band-limited signals exist (iterative Papoulis-Gerchberg)
  - alternatives for non-regularly sampled data (Lomb-Scargle)
  - alternatives for sparse/finite-rate-of-innovation signals
    - compressive sensing, sparse sampling

Digital signals

- Absolutely convergent sequences

$$(s[k])_{k\in\mathbb{Z}} \in l^1(\mathbb{R}) \Leftrightarrow \sum_{k=-\infty}^{\infty} |s[k]| < \infty$$

- Square-summable sequences

$$(s[k])_{k\in\mathbb{Z}} \in l^2(\mathbb{R}) \Leftrightarrow \sum_{k=-\infty}^{\infty} |s[k]|^2 < \infty$$

- Remark

$$l^1(\mathbb{R}) \subset l^2(\mathbb{R})$$

Discrete-Time Fourier transform (DTFT)

- Definition 1: z-transform

$$(s[k])_{k\in\mathbb{Z}} \xrightarrow{\text{ZT}} S(z) = \sum_{k=-\infty}^{\infty} s[k]z^{-k}$$

- Definition 2: Discrete-Time Fourier transform

$$(s[k])_{k\in\mathbb{Z}} \xrightarrow{\text{DTFT}} S(f) = \sum_{k=-\infty}^{\infty} s[k]z^{-k}, \quad z = e^{\imath 2\pi f}$$

Discrete-Time Fourier transform (DTFT)

- Normalized frequency

$$f = T f_{\text{phys}}$$

- Periodicity

$$f = 1 \ : S(f+1) = S(f)$$

- Existence

$$(s[k])_{k \in \mathbb{Z}} \in l^2(\mathbb{R})$$

- Special case: if $(s[k])_{k \in \mathbb{Z}} \in l^1(\mathbb{R})$, then $S(f)$ is continuous and bounded

DTFT - properties 1

- Linearity

$$\lambda s_1[k] + \mu s_2[k] \stackrel{\text{DTFT}}{\longrightarrow} \lambda S_1(f) + \mu S_2(f)$$

- Integer delay/translation

$$s[k-n] \stackrel{\text{DTFT}}{\longrightarrow} e^{-\imath 2\pi n f} S(f)$$

- Modulation

$$e^{\imath 2\pi\nu} s[k] \stackrel{\text{DTFT}}{\longrightarrow} S(f-\nu)$$

- Time inversion

$$s[-k] \stackrel{\text{DTFT}}{\longrightarrow} S(-f)$$

DTFT - properties 2

- Conjugaison

$$s^*[k] \stackrel{\text{DTFT}}{\longrightarrow} S^*(-f)$$

- Parseval-Plancherel equalities

$$\sum_{k=-\infty}^{\infty} s_1[k] s_2^*[k] = \int_{-1/2}^{1/2} S_1(f) S_2^*(f) df$$

$$\sum_{k=-\infty}^{\infty} |s[k]|^2 = \int_{-1/2}^{1/2} |S(f)|^2 df$$

DTFT - properties 3

- Convolution

$$(s_1 * s_2)[k] = \sum_{l=-\infty}^{\infty} s_1[l] s_2[k-l] \xrightarrow{\text{DTFT}} S_1(f) S_2(f)$$

- Inversion

$$s[k] = \int_{-1/2}^{1/2} S(f) e^{\imath 2\pi f k} df$$

Fourier series

- If s is periodic (and continuous), $s(t + 2\pi) = s(t)$ define:

$$a_k = \frac{1}{\pi} \int s(t) \cos(kt) dt$$

$$b_k = \frac{1}{\pi} \int s(t) \sin(kt) dt$$

  then the infinite Fourier series is:

$$S(k) = \frac{a_0}{2} + \sum_{k=1}^{\infty} [a_n \cos(kt) + b_n \sin(kt)]$$

Discrete Fourier transform

- Definition

$$(s[k])_{0\leq k\leq K-1} \xrightarrow{\mathrm{DTF_K}} \hat{s}[p]_{0\leq p\leq K-1}$$

with

$$\hat{s}[p]_{0\leq p\leq K-1} = \sum_{k=0}^{K-1} s[k]e^{-\imath 2\pi\frac{kp}{K}}$$

- Link to the DTFT : if $s[k] = 0$ pour $k < 0$ et $k \geq K$, then

$$\hat{s}[p] = S\left(\frac{p}{K}\right)$$

i.e. K-sample sampling of DTFT $S(f)$ on $[0, 1]$

- Inversion

$$s[k] = \frac{1}{K}\sum_{p=0}^{K-1} \hat{s}[p]e^{\imath 2\pi\frac{kp}{K}}$$

All Fourier transforms unite: Pontryagin duality

Transform	Original domain	Transform domain
Fourier transform	$\mathbb{R}$	$\mathbb{R}$
Discrete-time Fourier transform (DTFT)	$\mathbb{Z}$	T
Fourier series	T	$\mathbb{Z}$
Discrete Fourier transform (DFT)	$\mathbb{Z}/n\mathbb{Z}$	$\mathbb{Z}/n\mathbb{Z}$

All Fourier transforms unite: Pontryagin duality

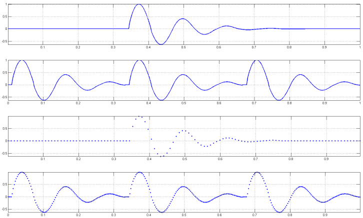


Figure 2: Signals on the different domains

Fast Fourier transform - FFT

- Fast algorithms exist (since Gauss)

$$\mathrm{FFT_K} \Rightarrow \text{complexity of } O(K\log_2(K)) \text{ operations}$$

- Cyclic or periodic convolution: Let $(s_1[k])_{0\leq k\leq K-1}$ and $(s_2[k])_{0\leq k\leq K-1}$

$$(s_1 \circledast s_2)[k] \xrightarrow{\mathrm{DTF_K}} \hat{s}_1[p]\cdot\hat{s}_2[p]$$

where $(s_1 \circledast s_2)[k]$ represents the K-point convolution of the periodized sequences:

$$(s_1 \circledast s_2)[k] = \sum_{l=0}^{k} s_1[l]s_2[k-l] + \sum_{l=k+1}^{K-1} s_1[l]s_2[K+k-l]$$

Reminders - Key message

- Nature of the data and the transform
  - Continuous and discrete natures ARE different
  - Generally stuff works
  - Intuition may be misleading (ex.: FFT on 8-sample signals, non proper windows)
  - Sometimes special care is needed: re-interpolation, pre-processing to avoid edge effects, instabilities, outliers

Windows

- Several uses
  - apodization, tapering (edges, jumps)
  - "stationarizing"
  - spectral estimation, filter design

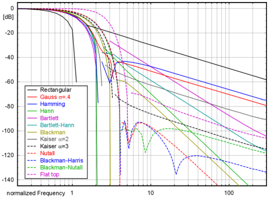


Figure 3: Origin: Wikipedia

- Many (parametric) designs:
  - Hann, Hamming, Kayser, Chebychev
  - generalized cosine: $A - B\cos(\frac{2\pi n}{M-1}) + C\cos(\frac{4\pi n}{M-1})$

Reminders: set averages

- $s(n)$: discrete time random process (stationary stochastic process)
- expectation:
$$\mu_s(n) = E\{s(n)\}$$
- variance:
$$\sigma_s^2(n) = E\{|s(n) - \mu_s(n)|^2\}$$
- autocovariance:
$$c_s(k,l) = E\{(s(k) - \mu_s(k))(s(l) - \mu_s(l))^*\}$$

Reminders: power spectral density

For an autocorrelation ergodic process:

$$\lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} s(n+k)s(n) = r_{ss}(k)$$

- if $s(n)$ is known for every n, power spectrum estimation
- caveat 1: samples are not unlimited $[0,\ldots,N-1]$, sometimes small
- caveat 2: corruption (noise, interfering signals)

Recast the problem: from the biased estimator of the ACF

$$\hat{r}_{ss}(k) = \sum_{n=0}^{N-1-k} s(n+k)s(n)$$

estimate power spectrum (periodogram)

$$\hat{P}_x(e^{\imath\omega}) = \sum_{k=-N+1}^{N-1} \hat{r}_{ss}(k)e^{\imath k\omega}$$

Time and frequency resolution

- Energy

$$E = \int |s(t)|^2 dt = \int |S(f)|^2 df$$

- Time or frequency location

$$\bar{t} = 1/E \int t|s(t)|^2 dt \qquad \overline{f} = 1/E \int f|S(f)|^2 df$$

- Energy dispersion

$$\Delta t = \sqrt{1/E \int (t-\bar{t})^2 |s(t)|^2 dt}$$

$$\Delta f = \sqrt{1/E \int (f-\overline{f})^2 |S(f)|^2 df}$$

Heisenberg-Gabor inequality

- Theorem (Weyl, 1931)
  If $s(t), ts(t), s'(t) \in \mathrm{L}^2$ then

$$\|s(t)\|^2 \leq 2\|ts(t)\|\|s'(t)\|$$

- Equality:
  Iff $s(t)$ is a modulated Gaussian/Gabor elementary function:

$$s'(t)/s(t) \propto t$$

$$s(t) = C \exp[-\alpha(t - t_m)^2 + \imath 2\pi\nu_m(t - t_m)]$$

- Proof
  Integration by part + Cauchy-Schwarz

Uncertainty principle (UP)

- Time-frequency UP
  For finite-energy every signal $s(t)$, with Δt and Δf finite:

$$\Delta t \Delta f \geq \frac{1}{4\pi}$$

  with equality for the modulated Gaussian only
- Principles

$$\|s'(t)\|^2 = |\imath 2\pi|^2 \|fS(f)\|^2$$

- Observations
  - Fourier (continuous) fundamental limit: arbitrary "location" cannot be attained both in time and frequency
  - have to choose between time and frequency locations
  - Gaussians are "the best"

Uncertainty principle (UP) for project management

Applies to other domains

Figure 4: Dilbert

Uncertainty principle - time

- One may write

$$s(t) = \int s(u)\delta(t-u)du$$

  - $\delta(t)$ is neutral w.r.t. convolution
  - interpreted as a decomposition of $s(t)$ onto a basis of shifted $\delta(t-u)$: $\Delta t = 0$ at
  - FT of basis functions: $e^{-\imath 2\pi f t}$: $\Delta f = \infty$

UP: as a limit of $0 \times \infty$

Uncertainty principle - frequency

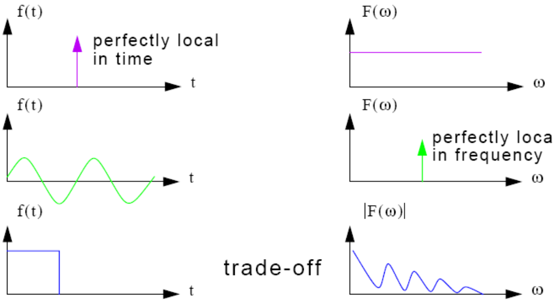
- One may write

$$s(t) = \int S(f) e^{\imath 2\pi f t} df$$

  - interpreted as a decomposition on pure waves $e^{\imath 2\pi f t}$: $\Delta t = \infty$
  - FT of basis functions: $\delta(f - t)$: $\Delta f = 0$

UP: as a limit of $\infty \times 0$

Uncertainty principle - illustration



Basis formalism interpretation

- Orthonormality

$$\langle e^{\imath 2\pi f t}, e^{\imath 2\pi g t} \rangle = \delta(f-g)$$

$$\langle \delta(f), \delta(g) \rangle = \delta(f-g)$$

- Scalar product

$$S(f) = \langle s(t), e^{\imath 2\pi f t} \rangle$$

$$s(t) = \langle s(t), \delta(f) \rangle$$

- Matching of a signal with a vector, a basis function (pure wave, Dirac)
  - Synthesis: continuous sum of orthogonal projection onto basis functions
  - Relative interest of the two bases? Other bases? (Walsh-Hadamard, DCT, eigenbase)
  - How to cope with mixed resolution?

Sliding window Fourier transform

- Principles
  Fourier analysis on time-space slices of the continuous $s(t)$ with a sliding window $h(t-\tau)$
- Short-term/short-time Fourier transform (STFT)

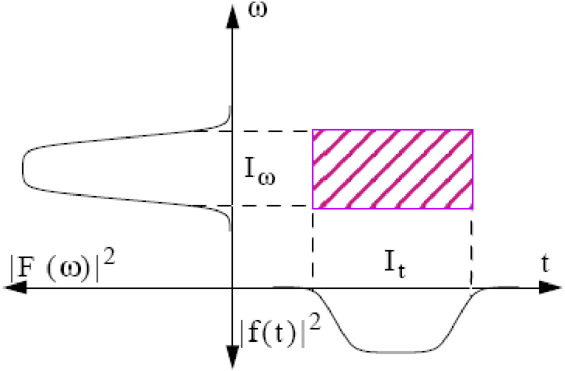
$$S_s(\tau, f; h) = \int s(t)h^*(t-\tau)e^{-\imath 2\pi f t}$$

- Wider domain of applications than FT
  - depend on h
  - FT as a peculiar instance (valid for other transforms: not a new tool, only a more versatile "leatherman"-like multi-tool)

Sliding window Fourier transform

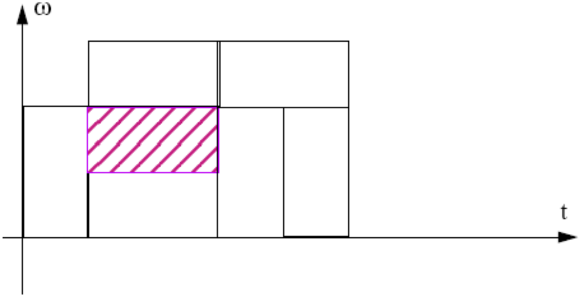
Figure 5: Leatherman wave black

Sliding window Fourier transform - illustration



Sliding window Fourier transform - time-freq. completude

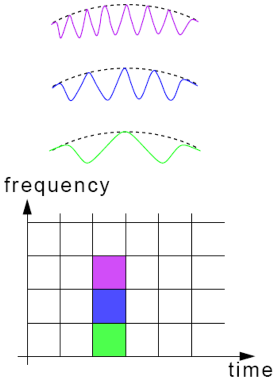
Notion of a "complete" descrition (*i.e.* somehow invertible)



Sliding window Fourier transform - windows

- Related to frequency analysis
  Depend on the window choice h (shape, length)
- Continuous time windows
  - rectangular: poor frequency resolution
  - gaussian: best time-frequency trade-off? (Gabor, 1946)
- Discrete time windows
  - τ discretized (jumps vs. redundancy)
  - different criteria: side lobes, *equiripple*, apodizing; Bartlett, Hamming, Hann, Blackmann-Harris, Blackmann-Nutall, Kaiser, Chebychev, Bessel, Generalized raised cosine, Lánczos, Flat-top,...

Sliding window Fourier transform - paving



Sliding window Fourier transform - reconstruction

- Simple analogy
  - synthesis: what two numbers add to result 3
  - $a + b = 3$: infinite number of solutions, *e.g.* $2945.75 + (-2942.75)$ but irrelevant
  - assume they are integers?
  - assume they are positive? $1 + 2 = 3$ or $2 + 1 = 3$
  - aim: increase interpretability, information compaction $(0 + 3 = 3)$, reduce overshoot

Sliding window Fourier transform - reconstruction

- Redundant transformation!
- Inversion

$$s(t) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} S_s(u,\xi;h)g(t-u)e^{\imath 2\pi t u}dud\xi,$$

provided that

$$\int_{-\infty}^{+\infty} g(t)h^*(t)dt = 1.$$

(perfect reconstruction, no information loss)

  - special case: admissible normalized window h

$$g(t) = h(t)$$

    but not the only solution (truncated $\sin$)
  - a bit more involved for discrete time, less if only approximate

Sliding window Fourier transform - spectrogram

- Definition

$$|S_s(\tau, f; h)|^2$$

- The spectrogram is a (bilinear) time-frequency distribution

$$E = \iint |S_s(\tau, f; h)|^2 d\tau df$$

for normalized admissible window h

- Parseval formula

$$\langle s_1, s_2 \rangle = \iint S_{s_1}(\tau, f; h) S_{s_2}(\tau, f; h) d\tau df$$

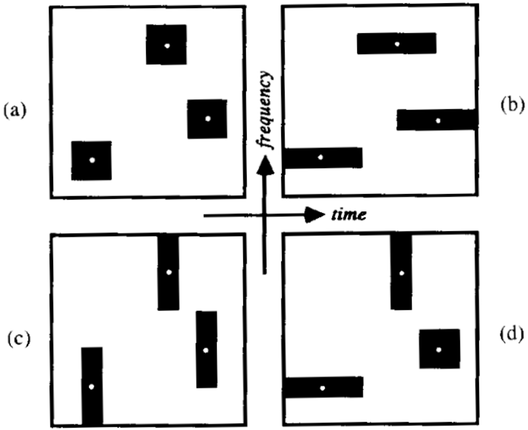
Sliding window Fourier transform - monoresolution

- Reason: basis functions

$$h(t-\tau)e^{\imath 2\pi f t}$$

  all possess similar resolution
- Examples:
  - $s(t) = \delta(t-t_0) \longrightarrow |S_s(\tau, f; h)|^2 = |h(t_0-\tau)|^2$
  - $s(t) = e^{\imath 2\pi f_0 t} \longrightarrow |S_s(\tau, f; h)|^2 = |H(f_0-f)|^2$
- Uses
  - for long range oscillatory signals, long windows are necessary
  - for short range transient, short windows needed
  - possibility to use several in parallel
  - incentive to use several ones simultaneously

Sliding window Fourier transform - illustration



Other time frequency distributions

- Quadratic or bilinear distributions
  - Wigner-Ville and avatars (smoothed, pseudo-, reweighted)
  - Cohen class (WV, Rihaczek, Born-Jordan, Choi-Williams)
  - property based: covariance, unitarity, inst. freq. & group delay, localization (for specific signals), support preservation, positivity, stability, interferences
  - Bertrand class, fractional Fourier transforms, linear canonical transformation (4 param.: FT, fFT, Laplace, Gauss-Weierstrass, Segal-Bargmann, Fresnel transforms)
  - generally not applied in more than 1-D